arXiv:math/0609395v1 [math.PR] 14 Sep 2006

# On random measures, unordered sums and discontinuities of the first kind

Frank Oertel
*Department of Mathematics*
*University College Cork*

By investigating in detail discontinuities of the first kind of real-valued functions and the analysis of unordered sums, where the summands are given by values of a positive real-valued function, we develop a measure-theoretical framework which in particular allows us to describe *rigorously* the representation and meaning of sums of jumps of type $\sum_{0<s\leq t} \Phi \circ |\Delta X_s|$, where $X : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a stochastic process with regulated trajectories, $t \in \mathbb{R}_+$ and $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a strictly increasing function which maps 0 to 0 (cf. Proposition 3.13). Moreover, our approach enables a natural extension of the jump measure of càdlàg and adapted processes to an integer-valued random measure of optional processes with regulated trajectories which need not necessarily to be right- or left-continuous (cf. Theorem 4.5). In doing so, we provide a detailed and constructive proof of the fact that the set of all discontinuities of the first kind of a given real-valued function on $\mathbb{R}$ is at most countable (cf. Lemma 2.3, Theorem 2.5 and Theorem 2.6).

By using the powerful analysis of unordered sums, we hope that our contributions fill an existing gap in the literature, since neither a detailed proof of (the frequently used) Theorem 2.5 nor a precise definition of sums of jumps seems to be available yet.

## 1. Preliminaries and notations

In this section, we introduce the basic notation and terminology which we will throughout in this paper. To perpetuate the lucidity of the main ideas, we only consider $\mathbb{R}$-valued functions and $\mathbb{R}$-valued trajectories of stochastic processes, although a transfer to the (finite) multi-dimensional case is easily possible. Most of our notations and definitions including those ones originating from the general theory of stochastic processes and stochastic analysis are standard. We refer the reader to the monographs [5], [8], [9], [10], [11], [12] and [17]. Concerning a basic introduction to the the powerful theory

*AMS 2000 subject classifications.* 28A05, 40G99, 60G05, 60G57

*Key Words and Phrases.* Regulated functions, unordered sums, at most countable sets, jumps, optional stochastic processes, stopping times, random measures

of unordered sums, we recommend the monographs [7] and [16]. Since at most countable unions of pairwise disjoint sets play an important role in this paper, we use a symbolic abbreviation. For example, if $A := \bigcup_{n=1}^{\infty} A_n$, where $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, we write shortly $A := \dot{\bigcup}_{n=1}^{\infty} A_n$.

Throughout this paper, $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ denotes a fixed probability space, together with a fixed filtration $\mathbf{F}$. Even if it is not explicitly emphasized, the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ always is supposed to satisfy the usual conditions[‡]. A real-valued (stochastic) process $X : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ (which may be identified with the family of random variables $(X_t)_{t\geq 0}$, where $X_t(\omega) := X(\omega, t)$) is called *adapted* (with respect to $\mathbf{F}$) if $X_t$ is $\mathscr{F}_t$-measurable for all $t \in \mathbb{R}_+$. $X$ is called *right-continuous* (respectively *left-continuous*) if for all $\omega \in \Omega$ the trajectory $X_\bullet(\omega) : \mathbb{R}_+ \longrightarrow \mathbb{R}, t \mapsto X_t(\omega)$ is a right-continuous (respectively left-continuous) real-valued function. If all trajectories of $X$ do have left-hand limits (respectively right-hand limits) everywhere on $\mathbb{R}_+$, $X_- = (X_{t-})_{t\geq 0}$ (respectively $X_+ = (X_{t+})_{t\geq 0}$) denotes the *left-hand* (respectively *right-hand*) *limit process*, where $X_{0-} := X_{0+}$ by convention. If all trajectories of $X$ do have left-hand limits and right-hand limits everywhere on $\mathbb{R}_+$, the *jump process* $\Delta X = (\Delta X_t)_{t\geq 0}$ is well-defined on $\Omega \times \mathbb{R}_+$. It is given by $\Delta X := X_+ - X_-$ (cf. also Section 2). A right-continuous process whose trajectories do have left limits everywhere on $\mathbb{R}_+$, is known as a *càdlàg* process. If $X$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable, $X$ is said to be *measurable*. $X$ is said to be *progressively measurable* (or simply *progressive*) if for each $t \geq 0$, its restriction $X|_{\Omega\times[0,t]}$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable. Obviously, every progressive process is measurable and (thanks to Fubini) adapted.

A random variable $T : \Omega \longrightarrow [0, \infty]$ is said to be a *stopping time* or *optional time* (with respect to $\mathbf{F}$) if for each $t \geq 0$, $\{T \leq t\} \in \mathscr{F}_t$. Let $\mathcal{T}$ denote the set of all stopping times, and let $S, T \in \mathcal{T}$ such that $S \leq T$. Then $[\![S, T[\![ := \{(\omega, t) \in \Omega \times \mathbb{R}_+ : S(\omega) \leq t < T(\omega)\}$ is an example for a *stochastic interval*. Similarly, one defines the stochastic intervals $]\!]S, T]\!]$, $]\!]S, T[\![$ and $[\![S, T]\!]$. Note again that $[\![T]\!] := [\![T, T]\!] = \mathrm{Gr}(T)|_{\Omega\times\mathbb{R}_+}$ is simply the graph of the stopping time $T : \Omega \longrightarrow [0, \infty]$ restricted to $\Omega \times \mathbb{R}_+$. $\mathcal{O} = \sigma\big\{[\![T, \infty[\![ : T \in \mathcal{T}\big\}$ denotes the *optional $\sigma$-field* which is generated by all càdlàg adapted processes. The *predictable $\sigma$-field* $\mathcal{P}$ is generated by all left-continuous adapted processes. An $\mathcal{O}$- (respectively $\mathcal{P}$-) measurable process is called *optional* or *well-measurable* (respectively *predictable*). All

[‡] $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets and $\mathbf{F}$ is right-continuous.

optional or predictable processes are adapted. For the convenience of the reader, we recall and summarise the precise relation between those different types of processes in the following

THEOREM 1.1 *Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space such that $\mathbf{F}$ satisfies the usual conditions. Let $X$ be a stochastic process on $\Omega \times \mathbb{R}_+$. Consider the following statements:*

(*i*) *$X$ is predictable;*

(*ii*) *$X$ is optional;*

(*iii*) *$X$ is progressive;*

(*iv*) *$X$ is adapted.*

*Then the following implications hold:*

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).$$

*If $X$ is right-continuous, then the following implications hold:*

$$(i) \Rightarrow (ii) \iff (iii) \iff (iv).$$

*If $X$ is left-continuous, then all statements are equivalent.*

*Proof.* The general chain of implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ is well-known (for a detailed discussion cf. e.g. [5], Chapter 3). If $X$ is left-continuous and adapted, then $X$ is predictable. Hence, in this case, all four statements are equivalent. If $X$ is right-continuous and adapted, then $X$ is optional (cf. [5], Remark following Theorem 3.4. and [8], Theorem 4.32). In particular, $X$ is progressive. □

By identifying processes that are almost everywhere identical, there is no difference between adapted *measurable* processes, optional processes, progressive processes and predictable processes (cf. [14]). In particular, since every adapted right-continuous process is optional, hence measurable, it is therefore almost everywhere identical to a predictable process.

Let $A \subseteq \Omega \times \mathbb{R}_+$ and $\omega \in \Omega$. Consider

$$D_A(\omega) := \inf\{t \in \mathbb{R}_+ : (\omega, t) \in A\} \in [0, \infty]$$

$D_A$ is said to be the *début* of $A$. Recall that $\inf(\emptyset) = +\infty$ by convention. $A$ is called a *progressive set* if $\mathbb{1}_A$ is a progressively measurable process. For a better understanding of the main ideas in the proof of Theorem 4.3, we need the following non-trivial result (a detailed proof of this statement can be found in e. g. [3] or [8]):

THEOREM 1.2 *Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space such that $\mathbf{F}$ satisfies the usual conditions. Let $A \subseteq \Omega \times \mathbb{R}_+$. If $A$ is a progressive set, then $D_A$ is a stopping time.*

## 2. Discontinuities of the first kind

In the following, let us denote by $I$ an arbitrary (bounded or non-bounded) closed interval in $\mathbb{R}$, containing at least two elements. In other words, let $I$ be precisely one of the following sets:

$$[a, b], [a, \infty), (-\infty, a], \mathbb{R},$$

where $a, b \in \mathbb{R}$, $a < b$. Let $f : I \longrightarrow \mathbb{R}$ be a real-valued function and $t \in I$ such that $(t, \infty) \cap I \neq \emptyset$.[§] Recall that the real value $f(t+)$ is the *right-hand limit of* $f$ at $t$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(t+)-f(s)| < \varepsilon$ whenever $s \in (t, t+\delta)$.[¶] Let $t \in I$ such that $I \cap (-\infty, t) \neq \emptyset$. The real value $f(t-)$ is said to be the *left-hand limit of* $f$ at $b$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(t-)-f(s)| < \varepsilon$ whenever $s \in (t-\delta, t)$. Let us denote by

$$L^+(f) := \{t \in I : (t, \infty) \cap I \neq \emptyset \text{ and } f(t+) \text{ exists}\}$$

the set of all finite right-hand limits of $f$, and by

$$L^-(f) := \{t \in I : I \cap (-\infty, t) \neq \emptyset \text{ and } f(t-) \text{ exists}\}$$

the set of all finite left-hand limits of $f$. Let $t \in L(f) := L^+(f) \cap L^-(f)$. Then $\Delta f(t) := f(t+) - f(t-) \in \mathbb{R}$ denotes the *jump of* $f$ *at* $t$, leading to the well-defined function $\Delta f : L(f) \longrightarrow \mathbb{R}$, the associated *function of jumps of* $f$. Let $\mathrm{int}(I)$ denote the interior of $I$. An easy calculation shows that

$$\mathrm{int}(I) = \{t \in I : (t, \infty) \cap I \neq \emptyset\} \cap \{t \in I : I \cap (-\infty, t) \neq \emptyset\},$$

[§] Any interior point of $I$ satisfies that condition.

[¶] Due to the choice of $t$ and the structure of $I$, we obviously may choose $\delta > 0$ sufficiently small such that $(t, t + \delta) \subseteq I$.

and it follows that

$$L(f) = \mathrm{int}(I) \cap \{t \in I : f(t-) \text{ exists and } f(t+) \text{ exists}\} \tag{2.1}$$

is a subset of $\mathrm{int}(I)$. The set $L(f)$ is known as the set of *discontinuities of the first kind* of $f$ or *jump points* of $f$. $I \setminus L(f)$ is called the set of *discontinuities of the second kind* of $f$ (cf. [11]).

Fix an arbitrary $\varepsilon > 0$ and consider the set $J(f;\varepsilon)$ of all jumps of $f$ of size at least $\varepsilon$, i. e.,

$$J(f;\varepsilon) := \{t \in L(f) : |\Delta f(t)| \geq \varepsilon\}.$$

The set of *all* jumps of the function $f$ is then given by

$$J(f) := \{t \in L(f) : \Delta f(t) \neq 0\} = \{t \in L(f) : |\Delta f(t)| > 0\} = \bigcup_{n\in\mathbb{N}} J(f; \frac{1}{n}).$$

Consider the function $\widetilde{f} : -I \longrightarrow \mathbb{R}$, defined by $\widetilde{f}(s) := f(-s)$. $\widetilde{f}$ simply describes the vertical reflection of $f$. Since the right-hand limit of $f$ (respectively the left-hand limit of $f$) is uniquely determined, vertical reflection of $f$ immediately implies the following important[‖]

PROPOSITION 2.1 *Let $f : I \longrightarrow \mathbb{R}$ be a real-valued function. Let $\widetilde{f} : -I \longrightarrow \mathbb{R}$, defined by $\widetilde{f}(s) := f(-s)$ for all $s \in -I$. Then*

*(i) $L^+(f) = -L^-(\widetilde{f})$, and $f(t+) = \widetilde{f}((-t)-)$ for all $t \in L^+(f)$;*

*(ii) $L^-(f) = -L^+(\widetilde{f})$, and $f(t-) = \widetilde{f}((-t)+)$ for all $t \in L^-(f)$.*

*In particular, $L(f) = -L(\widetilde{f})$ and*

$$J(f;\varepsilon) = -J(\widetilde{f};\varepsilon)$$

*for all $\varepsilon > 0$.*

Clearly, there exists a direct link to the well-known and rich class of regulated functions (cf. [7], 7.6. and [13]). By using our notation, recall that $f : I \longrightarrow \mathbb{R}$ is said to be *regulated on* $I$ if and only if if $\mathrm{int}(I) \subseteq \{t \in I : f(t-) \text{ exists and } f(t+) \text{ exists}\}$, if the left endpoint of $I$ belongs to $L^+(f)$, and if the right endpoint of $I$ belongs to $L^-(f)$ (if the latter exist). Consequently, due to (2.1), we may state the following

[‖]Note that the set $-I$ belongs to the same class as the given interval $I$.

Remark 2.2 Let $f : I \longrightarrow \mathbb{R}$ be a real-valued function. Then the following statements are equivalent:

(i) f is regulated on $I$;

(ii) $L(f) = \text{int}(I)$, the left endpoint of $I$ belongs to $L^+(f)$, and the right endpoint of $I$ belongs to $L^-(f)$ (if the latter exist).

Let $t \in \mathbb{R}$ and $(t_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of elements in a given non-empty subset $A$ of $\mathbb{R}$. If $\lim\limits_{n\to\infty} t_n = t$ and $t < t_{n+1} < t_n$ (respectively $t_n < t_{n+1} < t$) for all $n \in \mathbb{N}$, as usual, we make use of the shorthand notation $t_n \downarrow t$ (respectively $t_n \uparrow t$). Since compact intervals will play an important role later on, the next statement is given for $I := [a, b]$ only, where $a < b$. However, as the proof clearly shows, our arguments are of local nature, so that we actually may choose every interval $I$ of the above type (including $\mathbb{R}_+$).

Lemma 2.3 *Let $a < b$, $f : [a, b] \longrightarrow \mathbb{R}$ be an arbitrary real-valued function and $t \in [a, b]$.*

*(i) Let $(t_n)_{n\in\mathbb{N}} \subseteq L(f)$ such that $t_n \downarrow t$. If $t \in L^+(f)$, then*

$$\lim_{n\to\infty} f(t_n-) = f(t+) = \lim_{n\to\infty} f(t_n+).$$

*(ii) Let $(t_n)_{n\in\mathbb{N}} \subseteq L(f)$ such that $t_n \uparrow t$. If $t \in L^-(f)$, then*

$$\lim_{n\to\infty} f(t_n-) = f(t-) = \lim_{n\to\infty} f(t_n+).$$

*In each of these cases, we have*

$$\lim_{n\to\infty} |f(t_n+) - f(t_n-)| = 0.$$

*Proof.* To verify $(i)$, let $t \in L^+(f)$ and $(t_n)_{n\in\mathbb{N}} \subseteq L(f)$ such that $t_n \downarrow t$ and $n, m \in \mathbb{N}$ arbitrary. Put $\tau_{mn} := t_n - \xi_{mn}$, where $0 < \xi_{mn} := \frac{t_n - t_{n+1}}{2^m}$. Then

$$t < t_{n+1} < \tau_{mn} < t_n$$

for all $m, n \in \mathbb{N}$, and $\tau_{mn} \uparrow t_n$ (as $m \to \infty$) for all $n \in \mathbb{N}$. Thus, using the definition of left-hand limits, we have

$$f(t_n-) = \lim_{m\to\infty} f(\tau_{mn}) \tag{2.2}$$

for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Since by assumption $t \in L^+(f)$, there exists a $\delta > 0$ such that

$$f\big((t, t+\delta)\big) \subseteq \big(f(t+) - \varepsilon, f(t+) + \varepsilon\big).$$

Since $t_n \downarrow t$, it follows that $\lim_{n\to\infty} f(t_n) = f(t+)$ and that there exists $N_\delta \in \mathbb{N}$ such that $t_n - t < \delta$ for all $n \geq N_\delta$. Consequently, $\tau_{mn} \in (t, t+\delta)$ for all $m \in \mathbb{N}$ and $n \geq N_\delta$, implying that $|f(t+) - f(\tau_{mn})| < \varepsilon$ for all for all $m, n \geq N_\delta$. In other words, if $t \in L^+(f)$, then the double-sequence limit $\lim_{m,n\to\infty} f(\tau_{mn}) = f(t+)$ exists! Thanks to a further epsilon-delta argument, we therefore obtain

$$f(t+) = \lim_{n\to\infty} \big(\lim_{m\to\infty} f(\tau_{mn})\big) \overset{(2.2)}{=} \lim_{n\to\infty} f(t_n-).$$

Now we use the same method to approach each $t_n$ decreasingly from the right side. More precisely, let $m, n \in \mathbb{N}$ arbitrary and put $\rho_{mn} := t_n + \xi_{m,n-1}$, where $\xi_{m,0} := 0$ and $0 < \xi_{mn} := \frac{t_n - t_{n+1}}{2^m}$. Then

$$t < t_n < \rho_{mn} < t_{n-1}$$

for all $m \in \mathbb{N}, n \in \mathbb{N} \cap [2, \infty)$, and $\rho_{mn} \downarrow t_n$ (as $m \to \infty$) for all $n \in \mathbb{N}$. Thus, using the definition of right-hand limits, we have

$$f(t_n+) = \lim_{m\to\infty} f(\rho_{mn}) \tag{2.3}$$

for all $n \in \mathbb{N}$. Again, since $t \in L^+(f)$, we obtain the existence of a double-sequence limit, namely $\lim_{m,n\to\infty} f(\rho_{mn}) = f(t+)$. Hence,

$$f(t+) = \lim_{n\to\infty} \big(\lim_{m\to\infty} f(\rho_{mn})\big) \overset{(2.3)}{=} \lim_{n\to\infty} f(t_n+).$$

To complete the proof, we only have to consider the remaining case $(ii)$. So, let $t \in L^-(f)$ and $(t_n)_{n\in\mathbb{N}} \subseteq L(f)$ such that $t_n \uparrow t$. Then $s_n \downarrow s$, where $s_n := -t_n$ and $s := -t$. Consider the function $\widetilde{f} : [-b, -a] \longrightarrow \mathbb{R}$, defined by $\widetilde{f}(s) := f(-s)$. Due to Proposition 2.1, it follows that $s_n = -t_n \in -L(f) = L(\widetilde{f})$ for all $n \in \mathbb{N}$ and $s = -t \in -L^-(f) = L^+(\widetilde{f})$. Therefore, we precisely obtain the situation of part $(i)$, but now related to the function $\widetilde{f}$! Consequently,

$$\lim_{n\to\infty} \widetilde{f}(s_n-) = \widetilde{f}(s+) = \lim_{n\to\infty} \widetilde{f}(s_n+),$$

and the claim follows by Proposition 2.1. $\square$

If $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a regulated function, it follows that $L^+(f) = \mathbb{R}_+$ and $L^-(f) = (0, \infty)$. Hence, we may define $f_+(t) := f(t+) \in \mathbb{R}$ for all $t \in \mathbb{R}_+$ and $f_-(t) := f(t-) \in \mathbb{R}$ for all $t \in (0, \infty)$, implying the existence of well-defined functions $f_+ : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $f_- : (0, \infty) \longrightarrow \mathbb{R}$. A first immediate non-trivial implication of Lemma 2.3 is the following statement which will be used in the proof of Lemma 4.1.

Corollary 2.4 *Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a regulated function. Then $f_+$ is right-continuous on $\mathbb{R}_+$ and $f_-$ is left-continuous on $(0, \infty)$.*

Theorem 2.5 *Let $f : [a, b] \longrightarrow \mathbb{R}$ be an arbitrary real-valued function, where $a < b$. Then*

(*i*) *$J(f; \varepsilon)$ is finite for all $\varepsilon > 0$.*

(*ii*) *$J(f)$ is at most countable.*

*Proof.* Since $J(f) = \bigcup_{n \in \mathbb{N}} J(f; \frac{1}{n})$, we only have to prove (*i*). Assume by contradiction that $J(f; \varepsilon)$ is not finite. Due to the Bolzano-Weierstrass Theorem the bounded and infinite set $J(f; \varepsilon)$ has at least one accumulation point $t \in [a, b]$ (cf. e. g. [4]). Then there exists a sequence $(t_n)_{n \in \mathbb{N}} \subseteq J(f; \varepsilon)$ such that $t_n \to t$ (as $n \to \infty$), $t_k \neq t_l$ for all $k \neq l$, and $t_n \neq t$ for all $n \in \mathbb{N}$ (since $J(f; \varepsilon)$ is not finite). We therefore can select a monotone subsequence of $(t_n)_{n \in \mathbb{N}}$ which then also converges to $t$. To avoid some cumbersome notation, WLOG, we may assume that the original sequence $(t_n)_{n \in \mathbb{N}}$ is already the monotone one. Consequently, we arrived exactly at either scenario (*i*) or scenario (*ii*) of Lemma 2.3. Since $t_n \in J(f; \varepsilon)$ for all $n \in \mathbb{N}$, we clearly obtain a contradiction, and the claim follows. $\square$

The next result shows that at most countability of the jumps even can be guaranteed for all real-valued functions which are defined on the whole of $\mathbb{R}_+$ (respectively $\mathbb{R}$).

Theorem 2.6 *Let $f : J \longrightarrow \mathbb{R}$ be an arbitrary real-valued function, where $J \in \{\mathbb{R}_+, \mathbb{R}\}$. Then*

(*i*) *$J(f; \varepsilon)$ is at most countable for all $\varepsilon > 0$.*

(*ii*) *There exists a partition $\{D_k : k \in \mathbb{N}\}$ of $J(f)$ such that each $D_k$ is a finite subset of $J(f)$. In particular, $J(f)$ is at most countable.*

*Proof.* First, consider the case $J = \mathbb{R}_+$. Let $M := \{t \in \mathbb{R}_+ : f(t-) \text{ exists and } f(t+) \text{ exists}\}$. Since $\mathrm{int}(\mathbb{R}_+) = (0,\infty) = \dot\bigcup_{n=1}^{\infty}(n-1,n]$, representation (2.1) therefore implies that

$$L(f) = \dot{\bigcup_{n=1}^{\infty}} \big((n-1,n] \cap M\big) = \dot{\bigcup_{n=1}^{\infty}} \big((n-1,n) \cap M\big) \cup (\mathbb{N} \cap M) \overset{(2.1)}{=} \dot{\bigcup_{n=1}^{\infty}} L(f|_{[n-1,n]}) \cup (\mathbb{N} \cap M).$$

Hence,

$$J(f) = \dot{\bigcup_{n=1}^{\infty}} J(f|_{[n-1,n]}) \cup (\mathbb{N} \cap J(f)) \tag{2.4}$$

and

$$J(f;\varepsilon) = \dot{\bigcup_{n=1}^{\infty}} J(f|_{[n-1,n]};\varepsilon) \cup (\mathbb{N} \cap J(f;\varepsilon)) \tag{2.5}$$

for all $\varepsilon > 0$. Thus, (i) follows by Theorem 2.5. To prove (ii), fix $n \in \mathbb{N}$ and consider $f_n := f|_{[n-1,n]}$. Due to Theorem 2.5, the set $J(f_n;\frac{1}{m})$ is finite for each $m \in \mathbb{N}$. Since $J(f_n;\frac{1}{m}) \subseteq J(f_n;\frac{1}{m+1})$ for all $m \in \mathbb{N}$, it therefore follows that $J(f_n)$ can be written as an at most countable union of disjoint *finite* sets, namely as

$$J(f_n) = \bigcup_{m=1}^{\infty} J(f_n;\frac{1}{m}) = \dot{\bigcup_{m=1}^{\infty}} A_{m,n}, \tag{2.6}$$

where $A_{1,n} := J(f_n;1) = (\Delta f_n)^{-1}\big([1,\infty)\big)$ and $A_{m+1,n} := J(f_n;\frac{1}{m+1}) \setminus J(f_n;\frac{1}{m}) = (\Delta f_n)^{-1}\big([\frac{1}{m+1},\frac{1}{m})\big)$ for all $m \in \mathbb{N}$. Hence, (2.4) implies that

$$J(f) = \dot{\bigcup_{n=1}^{\infty}} \dot{\bigcup_{m=1}^{\infty}} A_{m,n} \cup (\mathbb{N} \cap J(f)) = \dot{\bigcup_{k=1}^{\infty}} B_k \cup (\mathbb{N} \cap J(f)),$$

where $\{B_k : k \in \mathbb{N}\} = \{A_{m,n} : (n,m) \in \mathbb{N} \times \mathbb{N}\}$. Consequently,

$$J(f) = \dot{\bigcup_{l=1}^{\infty}} D_l,$$

where $D_l := B_l \cup (\{l\} \cap J(f))$ is a finite set for all $l \in \mathbb{N}$. $\square$

Since partitions of this type will play a fundamental role, we introduce the following

Definition 2.7 *Let $D$ be an at most countable subset of $\mathbb{R}$ which is not empty. A partition $\{D_k : k \in \mathbb{N}\}$ of $D$ is called a finitely layered partition of $D$ if $D = \dot{\bigcup}_{k=1}^{\infty} D_k$, where $D_k$ is a finite subset of $D$ for all $k \in \mathbb{N}$.*

## 3. Unordered sums

Using the previous results about the structure of the sets $L(f)$ and $J(f)$, we can introduce sum of jumps functions like e. g. $\mathscr{P}(L(f)) \ni B \mapsto \sum_{s\in B}(\Delta f(s))^2 \in [0, \infty]$ in a mathematically concise manner. Our aim is to provide an exact description of such sums which is independent of the choice of the partition of $J(f)$ (cf. Theorem 3.8 and Proposition 3.13). In particular, we will show that finitely layered jump partitions provide a natural frame for integer valued random measures which are a *special case* of such a (randomised) sum (cf. Theorem 4.5).

To this end, let $L$ be an arbitrary non-empty set and $h : L \longrightarrow \mathbb{R}_+$ a *positive* real-valued function. Consider the set $\mathbb{F}(L) := \{F : F \text{ is a finite subset of } L\}$. Clearly, $(\mathbb{F}(L), \subseteq)$ is an ordered set, and we may therefore consider the well-defined net $s_h : \mathbb{F}(L) \longrightarrow \mathbb{R}_+$, defined by $s_h(F) := \sum_{s\in F} h(s)$, where $F \in \mathbb{F}(L)$. If the net $s_h$ converges to a limit point $p \in \mathbb{R}_+$, $\sum_{s\in L} h(s) := p$ is called the *unordered sum over $L$*. If the net $s_h$ converges, $s_h$ is called *summable*. Let us recall the following

Theorem 3.1 *Let $L$ be an arbitrary non-empty set and $h : L \longrightarrow \mathbb{R}_+$ a positive real-valued function. Then the following statement s are equivalent:*

(*i*) *$\sum_{s\in L} h(s)$ exists;*

(*ii*) *The set $\{s_h(F) : F \in \mathbb{F}(L)\}$ is bounded in $\mathbb{R}_+$.*

*If the net $s_h$ converges, then $\sum_{s\in L} h(s) = \sup\{s_h(F) : F \in \mathbb{F}(L)\}$.*

Since we have to include the case that the net $s_h$ is not convergent, Theorem 3.1 justifies the following natural extension of the unordered sum above:

Definition 3.2 *Let $L$ be an arbitrary non-empty set and $h : L \longrightarrow \mathbb{R}_+$ an arbitrary positive real-valued function. Define*

$$\sum_{s\in L} h(s) := \sup\Big\{\sum_{s\in F} h(s) : F \in \mathbb{F}(L)\Big\}$$

*If $\emptyset \neq A \subseteq L$, put $\sum_{s\in A} h(s) := \sum_{s\in A} h|_A(s)$. Put $\sum_{s\in\emptyset} h(s) := 0$.*

First note that in general, $\sum_{s\in L} h(s) \in [0,\infty]$ and that $\sum_{s\in E} h(s) \leq \sum_{s\in F} h(s)$ for all subsets $E \subseteq F \subseteq L$. If $L = \{s_1, \ldots, s_n\}$ itself is a finite set, then obviously $\sum_{s\in L} h(s) = \sum_{i=1}^{n} h(s_i) = \sum_{i=1}^{n} h(s_{\sigma(i)})$ for all permutations $\sigma \in S_n$, which justifies the notation. However, the following important fact, which we will use later on, requires a proof.

LEMMA 3.3 *Let $L$ be an arbitrary non-empty set and $h : L \longrightarrow \mathbb{R}_+$ an arbitrary positive real-valued function. Let $A$ and $B$ be arbitrary subsets of $L$. Then the following statements hold:*

*(i)* $\sum_{s\in L} h(s)\mathbb{1}_A(s) < +\infty$ *if and only if* $\sum_{s\in A} h(s) < +\infty$. *Moreover,*

$$\sum_{s\in A} h(s) = \sum_{s\in L} h(s)\mathbb{1}_A(s).$$

*(ii)* $\sum_{s\in A} h(s)\mathbb{1}_B(s) < +\infty$ *if and only if* $\sum_{s\in A\cap B} h(s) < +\infty$. *Moreover,*

$$\sum_{s\in A\cap B} h(s) = \sum_{s\in A} h(s)\mathbb{1}_B(s).$$

*These sums may be finite or infinite.*

*Proof.* Since (ii) obviously follows by (i) (by applying (i) to the function $h\mathbb{1}_B$), we only have to prove (i). If $A = \emptyset$, nothing is to prove. So, let $A \neq \emptyset$. Assume first that $\sum_{s\in A} h(s) < +\infty$. Let $F$ be an arbitrary finite subset of $L$. Since $F$ equals the disjoint union of the (finite) sets $A\cap F$ and $(L\setminus A)\cap F$, standard associative and commutative summation of finitely many numbers immediately gives

$$\sum_{s\in F} h(s)\mathbb{1}_A(s) = \sum_{s\in A\cap F} h(s)\mathbb{1}_A(s) = \sum_{s\in A\cap F} h(s).$$

Since the finite subset $F$ of $L$ was arbitrarily chosen, it therefore follows that

$$\sum_{s\in L} h(s)\mathbb{1}_A(s) \leq \sum_{s\in A} h(s) < +\infty.$$

Now let $\sum_{s\in L} h(s)\mathbb{1}_A(s)$ be finite. Then, if $G$ is an arbitrary finite subset of $A \subseteq L$, we obviously have

$$\sum_{s\in G} h(s) = \sum_{s\in G} h(s)\mathbb{1}_A(s) \leq \sum_{s\in L} h(s)\mathbb{1}_A(s) < +\infty,$$

which proves the other inequality. Consequently, we have shown that the equality holds if $\sum_{s\in A} h(s)$ is finite or if $\sum_{s\in L} h(s)\mathbb{1}_A(s)$ is finite. Hence, it must be true if these sums are finite or if this is not the case. $\square$

PROPOSITION 3.4 *Let $L$ be an arbitrary non-empty set, $\alpha, \beta \geq 0$ and $h, g : L \longrightarrow \mathbb{R}_+$ arbitrary positive real-valued functions. Then $\sum_{s\in L} g(s) < +\infty$ and $\sum_{s\in L} h(s) < +\infty$ if and only if $\sum_{s\in L}(\alpha g(s) + \beta h(s)) < +\infty$. Moreover,*

$$\sum_{s\in L}(\alpha g(s) + \beta h(s)) = \alpha \sum_{s\in L} g(s) + \beta \sum_{s\in L} h(s).$$

*These sums may be finite or infinite.*

*Proof.* First, let $\sum_{s\in L} g(s) < +\infty$ and $\sum_{s\in L} h(s) < +\infty$. Since addition is associative and commutative, the equality obviously is true for every finite subset $F$ of $L$. Consequently, we already obtain the inequality

$$\sum_{s\in L}(\alpha g(s) + \beta h(s)) \leq \alpha \sum_{s\in L} g(s) + \beta \sum_{s\in L} h(s) < +\infty.$$

Now let $\sum_{s\in L}(\alpha g(s) + \beta h(s)) < +\infty$. Let $E$ be an arbitrary finite subset of $L$. Then,

$$\max\left\{\alpha \sum_{s\in E} g(s), \beta \sum_{s\in E} g(s)\right\} \leq \sum_{s\in E}(\alpha g(s)+\beta h(s)) \leq \sum_{s\in L}(\alpha g(s)+\beta h(s)) < +\infty,$$

and it follows that both, $\Gamma := \sum_{s\in L} g(s)$ and $\Delta := \sum_{s\in L} h(s)$ are finite. To prove the other inequality, let $\varepsilon > 0$ be given. Then there exist finite subsets $F$ and $G$ of $L$ such that

$$\alpha\Gamma + \beta\Delta < \sum_{s\in F} \alpha g(s) + \sum_{s\in G} \beta h(s) + \varepsilon.$$

Since we currently are working with summation of finitely many elements only, we obviously may conclude that

$$\alpha\Gamma + \beta\Delta - \varepsilon < \sum_{s\in F\cup G} \alpha g(s) + \sum_{s\in F\cup G} \beta h(s) = \sum_{s\in F\cup G} (\alpha g(s) + \beta h(s)).$$

Since $F \cup G$ is a finite subset of $L$, we have arrived at the other inequality. Hence, similarly as in the proof of Lemma 3.3, we have shown that the equality holds if $\sum_{s\in L}(\alpha g(s) + \beta h(s)) < +\infty$ or if $\sum_{s\in L} g(s) < +\infty$ and $\sum_{s\in L} h(s) < +\infty$. $\square$

Corollary 3.5 *Let $L$ be an arbitrary non-empty set and $h : L \longrightarrow \mathbb{R}_+$ an arbitrary positive real-valued function. Let $C$ and $D$ be arbitrary subsets of $L$. If $C \cap D = \emptyset$, then*

$$\sum_{s \in C \cup D} h(s) = \sum_{s \in C} h(s) + \sum_{s \in D} h(s).$$

*These sums may be finite or infinite.*

*Proof.* Since $C \cap D = \emptyset$, we have $\mathbb{1}_{C \cup D} = \mathbb{1}_C + \mathbb{1}_D$. Hence, by Lemma 3.3 and Proposition 3.4, it therefore follows that

$$\sum_{s \in C \cup D} h(s) = \sum_{s \in L} h(s)\mathbb{1}_{C \cup D}(s) = \sum_{s \in L}(h(s)\mathbb{1}_C(s)+h(s)\mathbb{1}_D(s)) = \sum_{s \in C} h(s)+\sum_{s \in D} h(s).$$

$\square$

Theorem 3.6 *Let $L$ be an arbitrary non-empty set and $h : L \longrightarrow \mathbb{R}_+$ a real-valued function. Let $\{D_n : n \in \mathbb{N}\}$ be an arbitrary partition of a set $D \subseteq L$. Then the following statements are equivalent:*

*(i)* $\sum_{s \in D} h(s) < +\infty$;

*(ii)* $\sum_{s \in D_n} h(s) < +\infty$ *for all* $n \in \mathbb{N}$ *and* $\Big(\sum_{s \in D_n} h(s)\Big)_{n \in \mathbb{N}} \in l_1$.

*Moreover,*

$$\sum_{s \in D} h(s) = \sum_{n=1}^{\infty} \sum_{s \in D_n} h(s).$$

*These sums may be finite or infinite.*

*Proof.* Nothing is to show if $D = \emptyset$. So, let $D \neq \emptyset$, and assume first that (i) holds. Since $D_n \subseteq D$ for all $n \in \mathbb{N}$, each finite subset of each $D_n$ is already a finite subset of $D$, implying that $\sum_{s \in D_n} h(s) \leq \sum_{s \in D} h(s) < +\infty$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ arbitrary and consider the set $C_n := \dot\bigcup_{k=1}^{n} D_k \subseteq D$. Due to Corollary 3.5, we have

$$0 \leq \sum_{k=1}^{n} \sum_{s \in D_k} h(s) = \sum_{s \in C_n} h(s) \leq \sum_{s \in D} h(s) < +\infty.$$

Since $n \in \mathbb{N}$ was arbitrarily chosen, we may conclude that

$$0 \leq \sum_{k=1}^{\infty} \sum_{s \in D_k} h(s) \leq \sum_{s \in D} h(s) < +\infty.$$

Hence, $\left( \sum_{s \in D_k} h(s) \right)_{k \in \mathbb{N}} \in l_1$, and statement (ii) follows. Now assume that (ii) holds. Then $0 \leq \sum_{s \in D_n} h(s) < +\infty$ for all $n \in \mathbb{N}$ and $0 \leq \sum_{n=1}^{\infty} \sum_{s \in D_n} h(s) < +\infty$. Let F be an arbitrary finite subset of $D$. Choose a sufficiently large number $n \in \mathbb{N}$ such that $F \subseteq \dot{\bigcup}_{k=1}^{n} D_k$, implying that $F = \dot{\bigcup}_{k=1}^{n} F_k$, where $F_k := F \cap D_k$ for all $k \in \{1, 2, \cdots, n\}$. Consequently, we have

$$\sum_{s \in F} h(s) = \sum_{k=1}^{n} \sum_{s \in F_k} h(s).$$

Since each $F_k$ is a finite subset of $D_k$, assumption (ii) further implies that

$$\sum_{k=1}^{n} \sum_{s \in F_k} h(s) \leq \sum_{k=1}^{n} \sum_{s \in D_k} h(s) \leq \sum_{k=1}^{\infty} \sum_{s \in D_k} h(s) < +\infty.$$

Since the finite subset $F$ of $D$ was arbitrarily chosen, it follows that statement (i) is true, and we have

$$\sum_{s \in D} h(s) \leq \sum_{n=1}^{\infty} \sum_{s \in D_n} h(s) < +\infty.$$

Clearly, we have shown that the equality holds if the case (i) or the case (ii) is given. Since (i) is equivalent to (ii), the equality necessarily also must hold if one of the both unordered sums is not finite. □

Since $\mathbb{N} \times \mathbb{N} = \dot{\bigcup}_{m=1}^{\infty} \dot{\bigcup}_{n=1}^{\infty} \{(m,n)\} = \dot{\bigcup}_{n=1}^{\infty} \dot{\bigcup}_{m=1}^{\infty} \{(m,n)\}$, Theorem 3.6 immediately recovers a well-known result concerning the rearrangement of the terms in a double series (cf. e. g. [4]):

Corollary 3.7 *Let $(a_{mn})_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ be an arbitrary double-sequence in $\mathbb{R}_+$. Then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = \sum_{n \in \mathbb{N} \times \mathbb{N}} a_{mn} = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}$$

By using the language of measure theory, we have proven the following important result:

Theorem 3.8 *Let $L$ be an arbitrary non-empty set and $h : L \longrightarrow \mathbb{R}_+$ an arbitrary positive real-valued function. Then*

$$\begin{array}{rcl} \mu_h : \mathscr{P}(L) & \longrightarrow & [0,\infty] \\ A & \mapsto & \displaystyle\sum_{s\in A} h(s), \end{array}$$

*is a well-defined measure on the measurable space $(L, \mathscr{P}(L))$.*

Remark 3.9 Let $A$ and $B$ be arbitrary subsets of $L$. Then Lemma 3.3 implies that

$$\mu_h(A) = \sum_{s\in L} h(s)\mathbb{1}_A(s) = \mu_{h\mathbb{1}_A}(L) \tag{3.1}$$

and

$$\mu_h(A \cap B) = \sum_{s\in A} h(s)\mathbb{1}_B(s) = \mu_{h\mathbb{1}_B}(A). \tag{3.2}$$

Dependent on the choice of the function $h$, we recognise two special and well-known cases:

(i) If $h(s) := 1 = \mathbb{1}_L(s)$ for all $s \in L$, then $\mu_{\mathbb{1}_L}$ is precisely the *counting measure*.

(ii) If $h := \mathbb{1}_{\{s_0\}}$, where $s_0 \in L$, we obtain exactly the *Dirac measure* at $s_0$, since

$$\mu_{\mathbb{1}_{\{s_0\}}}(A) \overset{(3.2)}{=} \mu_{\mathbb{1}_L}(\{s_0\} \cap A) \overset{(3.2)}{=} \mu_{\mathbb{1}_A}(\{s_0\}) = \mathbb{1}_A(s_0) = \delta_{s_0}(A)$$

for all $A \in \mathscr{P}(L)$.

Corollary 3.10 *Let $L$ be an arbitrary non-empty set and $h : L \longrightarrow \mathbb{R}_+$ an arbitrary positive real-valued function.*

*(i) If $A \in \mathscr{P}(L)$ is finite, then*

$$\mu_h(A) = \sum_{\nu=1}^{n} h(\nu),$$

*where $n = card(A)$.*

*(ii) If $A \in \mathscr{P}(L)$ is countable, then*

$$\mu_h(A) = \sum_{n=1}^{\infty} h(\varphi(n))$$

*for all bijective mappings $\varphi : \mathbb{N} \longrightarrow A$.*

*Proof.* First note that $\mu_h(\{a\}) = h(a)$ for all $a \in A \subseteq L$. Statement (i) now follows directly by Theorem 3.8. To prove (ii), let $a_n := \varphi(n)$, where $n \in \mathbb{N}$ is arbitrary. Then, by Theorem 3.8 again, we have

$$\mu_h(A) = \mu_h(\dot{\bigcup_{n=1}^{\infty}}\{a_n\}) = \sum_{n=1}^{\infty} \mu_h(\{a_n\}) = \sum_{n=1}^{\infty} h(a_n),$$

and the proof is finished. $\square$

We have developed all necessary tools which now allow us to give a lucid and short proof of the following non-trivial result.

THEOREM 3.11 *Let $L$ be an arbitrary non-empty set and $h : L \longrightarrow \mathbb{R}_+$ an arbitrary positive real-valued function. Put $P := \{s \in L : h(s) > 0\}$. If $\sum_{s \in L} h(s) < +\infty$, then $P$ is at most countable, and*

$$\sum_{s \in L} h(s) = \sum_{s \in P} h(s) = \sum_{n=1}^{\infty} h(\varphi(n))$$

*for all bijective mappings $\varphi : \mathbb{N} \longrightarrow P$.*

*Proof.* By assumption, $\Sigma := \sum_{s \in L} h(s) < +\infty$. Since $P = \bigcup_{n=1}^{\infty} P_n$, where $P_n := \{s \in L : h(s) > \frac{1}{n}\}$, we only have to show that each subset $P_n$ of $L$ is at most countable. We even show more and claim that

$$P_n \text{ is finite and consists of at most } \lfloor n\Sigma \rfloor \text{ elements for all } n \in \mathbb{N}, \qquad (3.3)$$

where $\mathbb{R} \ni x \mapsto \lfloor x \rfloor := \max\{m \in \mathbb{Z} : m \leq x\}$ describes the assignment rule of the floor function. We assume by contradiction that (3.3) is false. Then there would exist $m \in \mathbb{N}$ and a *finite* subset $G_m$ of $P_m$ such that $\text{card}(G_m) = \lfloor m\Sigma \rfloor + 1$. But then, due to the definition of the floor function, we would have

$$+\infty > \lfloor m\Sigma \rfloor + 1 > m\Sigma = \sum_{s \in L} mh(s) \geq \sum_{s \in G_m} mh(s) > \text{card}(G_m) \cdot 1 = \lfloor m\Sigma \rfloor + 1,$$

which obviously is a contradiction. Hence, statement (3.3) is true, implying that the set $P$ is at most countable.

Clearly, we have $\sum_{s \in L \setminus P} h(s) = 0$. Hence, $\sum_{s \in L} h(s) = \sum_{s \in P} h(s) = \mu_h(P)$ (due to Corollary 3.5), and Corollary 3.10 finishes the proof. $\square$

By linking Lemma 3.3 and (the proof of) Theorem 3.6, we can characterise the finiteness of the measure $\mu_h$ in the following way:

Proposition 3.12 *Let $L$ be an arbitrary non-empty set and $h : L \longrightarrow \mathbb{R}_+$ an arbitrary positive real-valued function. Then the following statements are equivalent:*

*(i)* $\mu_h : \mathscr{P}(L) \longrightarrow \mathbb{R}_+$ *is a finite measure.*

*(ii)* *If $\{L_n : n \in \mathbb{N}\}$ is an arbitrary partition of $L$ such that $\sum_{s \in L_n} h(s) < +\infty$ for all $n \in \mathbb{N}$ and $\left(\sum_{s \in L_n} h(s)\right)_{n \in \mathbb{N}} \in l_1$, then*

$$\mu_h(A) = \sum_{n=1}^{\infty} \sum_{s \in L_n} h(s) \mathbb{1}_A(s)$$

*for all $A \in \mathscr{P}(L)$.*

*(iii)* *There exists a partition $\{C_l : l \in \mathbb{N}\}$ of $L$ such that $\sum_{s \in C_l} h(s) < +\infty$ for all $l \in \mathbb{N}$, $\left(\sum_{s \in C_l} h(s)\right)_{l \in \mathbb{N}} \in l_1$ and*

$$\mu_h(A) = \sum_{l=1}^{\infty} \sum_{s \in C_l} h(s) \mathbb{1}_A(s)$$

*for all $A \in \mathscr{P}(L)$.*

We have arrived at a point now, where we can apply our general framework to discontinuities of the first kind. In particular, we can easily provide a representation of unordered sums over all jumps of $f$; a fact, which frequently is used in the literature on general semimartingales including references on Lévy processes, but which seemingly hasn't been *rigorously* proven yet, very similar to the case of the proof of Theorem 2.5 (cf. e.g. [1], [11], [15]).

PROPOSITION 3.13 *Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be an arbitrary function, and assume that $\emptyset \neq J(f)$. Let $\{D_n : k \in \mathbb{N}\}$ be an arbitrary partition of $J(f)$. Let $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be strictly increasing and continuous such that $\Phi(0) = 0$. Let $B$ be a non-empty subset of $L(f)$. Then*

$$\sum_{s\in B}\Phi\big(|\Delta f(s)|\big) = \mu_{\Phi\circ|\Delta f|}\big(B\cap J(f)\big) = \sum_{n=1}^{\infty}\sum_{s\in D_n}\Phi\big(|\Delta f(s)|\big)\mathbb{1}_B(s) = \sum_{n=1}^{\infty}\Phi\big(|\Delta f(\varphi(n))|\big)$$

*for all bijective mappings $\varphi : \mathbb{N} \longrightarrow B \cap J(f)$. If in addition $f$ is regulated, then*

$$\sum_{0<s\leq t}\Phi\big(|\Delta f(s)|\big) = \mu_{\Phi\circ|\Delta f|}\big((0,t]\cap J(f)\big) = \sum_{n=1}^{\infty}\sum_{s\in D_n}\Phi\big(|\Delta f(s)|\big)\mathbb{1}_{(a,t)}(s) = \sum_{n=1}^{\infty}\Phi\big(|\Delta f(\varphi(n))|\big)$$

*for all $t \in (0,\infty)$ and bijective mappings $\varphi : \mathbb{N} \longrightarrow (0,t] \cap J(f)$.*

*Proof.* Let $B$ be an arbitrary non-empty subset of $L(f)$. Due to Theorem 2.6, $D := J(f)$ is at most countable. Since $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly increasing and continuous, it is invertible, and $\Phi^{-1} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly increasing as well (due to the Inverse Function Theorem). Hence, the set $\{s \in B : \Phi(|\Delta f(s)|) > 0\} = \{s \in B : |\Delta f(s)| > 0\} = B \cap D$ is an at most countable subset of $D \subseteq L(f)$. Consider the the function $h := \Phi \circ |\Delta f|$. Then $\mu_h(B \cap D) \overset{(3.2)}{=} \mu_{h\mathbb{1}_B}(D)$. Since $\Phi(0) = 0$, equality (3.2) implies that $\mu_h(B \cap (L \setminus D)) = \mu_{h\mathbb{1}_B}(L \setminus D) = 0$, and it follows that

$$\sum_{s\in B}h(s) = \mu_h(B) = \mu_h(B\cap D) = \mu_{h\mathbb{1}_B}(D) = \sum_{n=1}^{\infty}\sum_{s\in D_n}h(s)\mathbb{1}_B(s).$$

Since the set $B\cap D$ is an at most countable subset of $L(f)$, the first statement follows by Corollary 3.10. If in addition $f$ is regulated, then $L(f) = (0,\infty)$ (due to Remark 2.2), implying that $B := (0,t] \subseteq L(f)$ for all $t \in (0,\infty)$. Now, the second statement follows immediately from the first one. $\square$

If $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ were regulated, a natural question would be to ask for the representation of the function of jumps $\Delta g : L(g) \longrightarrow \mathbb{R}$, where $g(t) := \sum_{s\in(0,t]}\Phi\big(|\Delta f(s)|\big)$, $t \in (0,\infty)$. To this end, let $h : (0,\infty) \longrightarrow \mathbb{R}_+$ be an arbitrary positive real-valued function, and assume that $\mu_h((0,\infty)) = \sum_{s\in(0,\infty)}h(s) < \infty$. Then $g(t) := \sum_{s\in(0,t]}h(s) = \mu_h((0,t]) \cap P) < +\infty$ for

all $t \in \mathbb{R}_+$, where $P := \{s \in (0,\infty) : h(s) > 0\}$. Let $t \in (0,\infty)$. Since $\mu_h : \mathscr{P}((0,\infty)) \longrightarrow \mathbb{R}_+$ is a (finite) measure, it follows that $g(t+\frac{1}{n}) - g(t-\frac{1}{n}) = \mu_h(I_n)$ for sufficiently large $n \in \mathbb{N}$, where $I_n := (t-\frac{1}{n}, t+\frac{1}{n}]$. Since $I_n \downarrow \{t\}$ as $n \to \infty$, we obviously have

$$\Delta g(t) = \lim_{n\to\infty} \mu_h(I_n) = \mu_h(\{t\}) = h(t).$$

Hence, $L(g) = (0,\infty)$, and $\Delta\Big(\sum_{s\in(0,\cdot)} h(s)\Big) = \Delta g = h$ on $(0,\infty)$. Moreover, since $g(\frac{1}{n}) = \mu_h((0,\frac{1}{n}]) \to \mu_h(\emptyset) = 0$ as $n \to \infty$, it follows that $0 \in L^+(g)$ and $g(0+) = 0 = g(0)$. Consequently, Remark 2.2 implies the following

Proposition 3.14 *Let $h : (0,\infty) \longrightarrow \mathbb{R}_+$ be an arbitrary positive real-valued function. If $\sum_{s\in(0,\infty)} h(s) < \infty$, then the function*

$$\begin{aligned} g : \mathbb{R}_+ &\longrightarrow \mathbb{R}_+ \\ t &\mapsto \sum_{s\in(0,t]} h(s), \end{aligned}$$

*is regulated, $L(g) = (0,\infty)$, and*

$$\Delta g = h.$$

## 4. Random measures induced by optional processes

Next, we transfer the main results of our previous investigations to (trajectories of) stochastic processes. Again, let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a given filtered probability space such that $\mathbf{F}$ satisfies the usual conditions. If $X$ is an adapted and càdlàg process, then it is well-known that the left limit process $X_-$ is predictable. Recall that every adapted and right-continuous process is optional (cf. Theorem 1.1). Consequently, we deal with a special case of the slightly more general

Lemma 4.1 *Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space such that $\mathbf{F}$ satisfies the usual conditions. Let $X : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a stochastic process such that all trajectories of $X$ are regulated. Then all trajectories of the left limit process $X_-$ (respectively of the right limit process $X_+$) are left-continuous (respectively right-continuous). If in addition $X$ is optional, then $X_-$ is predictable.*

*Proof.* Fix $\omega \in \Omega$ and consider the (fixed) trajectory $f := X_\bullet(\omega) : \mathbb{R}_+ \longrightarrow \mathbb{R}$. Since $f$ is a regulated function, it follows that that $L^-(f) = (0, \infty)$. Consequently, due to Corollary 2.4, it clearly follows that the trajectory $Y_\bullet(\omega)$ of the left limit process $Y := X_-$ is left-continuous on $(0, \infty)$. Similarly, it follows that the trajectory of the right limit process $X_+$ is right-continuous on $\mathbb{R}_+ = L^+(f)$. Now assume that in addition $X$ is optional and therefore adapted. Then $X_-$ is an adapted process too. Consequently it follows that $Y = X_-$ is adapted and left-continuous, and the definition of predictability finishes the proof. $\square$

Now, we return to *finitely* layered partitions and start with the following observation. Despite its seemingly clear context, it will be of high importance for our further investigations.

LEMMA 4.2 *Let $\emptyset \neq D$ be a finite subset of $\mathbb{R}$, consisting of $\kappa_D$ elements. Consider*

$$s_1^D := \min(D)$$

*and, if $\kappa_D \geq 2$,*

$$s_{n+1}^D := \min\bigl(D \cap (s_n^D, \infty)\bigr),$$

*where $n \in \{1, 2, \ldots, \kappa_D - 1\}$. Then $D \cap (s_n^D, \infty) \neq \emptyset$ for all $n \in \{1, 2, \ldots, \kappa_D - 1\}$ and $s_n^D < s_{n+1}^D$ for all $n \in \{1, 2, \ldots, \kappa_D - 1\}$. Moreover, we have*

$$D = \dot{\bigcup_{n=1}^{\kappa_D}} \{s_n^D\}.$$

*Proof.* Obviously, nothing is to prove if $\kappa_D \in \{1, 2\}$. Let $\kappa_D \geq 3$. Obviously, we have $D \cap (s_1^D, \infty) \neq \emptyset$. Now assume by contradiction that there exists $n \in \{2, \ldots, \kappa_D - 1\}$ such that $D \cap (s_n^D, \infty) = \emptyset$. Choose the *minimal* $m \in \{2, \ldots, \kappa_D - 1\}$ such that $D \cap (s_m^D, \infty) = \emptyset$. Then $s_k^D := \min\bigl(D \cap (s_{k-1}^D, \infty)\bigr) \in D$ is well-defined for all $k \in \{2, \ldots, m\}$, and we obviously have $s_1^D < s_2^D < \ldots < s_m^D$. Moreover, by construction of $m$, it follows that

$$s \leq s_m^D \text{ for all } s \in D. \tag{4.1}$$

Assume now that there exists $s^* \in D$ such that $s^* \notin \{s_1^D, s_2^D, \ldots, s_m^D\}$. Then, by (4.1), there must exist $l \in \{1, 2, \ldots, m - 1\}$ such that $s_l^D < s^* < s_{l+1}^D$, which is a contradiction, due to the definition of $s_{l+1}^D$. Hence, such a value $s^*$ cannot exist, and it consequently follows that $D = \{s_1^D, s_2^D, \ldots, s_m^D\}$. But

then $m = \mathrm{card}(D) \le \kappa_D - 1 < \kappa_D$, which is a contradiction. Hence, $s_k^D \in D$ is well-defined for all $k \in \{1, 2, \ldots, \kappa_D\}$. Since $\mathrm{card}(D) = \kappa_D$, the proof is finished. $\square$

THEOREM 4.3 *Let* $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ *be a filtered probability space such that* $\mathbf{F}$ *satisfies the usual conditions. Let* $X : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ *be an optional process such that all trajectories of* $X$ *are regulated. Then* $\Delta X$ *is also optional. Put* $X_{0-} := X_{0+}$. *If for each trajectory of* $X$ *its set of jumps is not finite, then there exists a sequence of stopping times* $(T_n)_{n\in\mathbb{N}}$ *such that* $(T_n(\omega))_{n\in\mathbb{N}}$ *is a strictly increasing sequence in* $(0, \infty)$ *for all* $\omega \in \Omega$ *and*

$$J(X_\bullet(\omega)) = \dot{\bigcup_{n=1}^{\infty}} \{T_n(\omega)\} \text{ for all } \omega \in \Omega,$$

*or equivalently,*

$$\{\Delta X \neq 0\} = \dot{\bigcup_{n=1}^{\infty}} [\![T_n]\!].$$

*Proof.* Since the filtration is right-continuous, a direct calculation shows that the right limit process $X_+$ is adapted. Due to Lemma 4.1, all paths of $X_+$ are right-continuous on $\mathbb{R}_+$. Since the filtration is right-continuous, it follows that $X_+$ is also adapted and hence an optional process. Consequently, since the process $X$ was assumed to be optional, and since each predictable process is optional, a further application of Lemma 4.1 implies that the jump process $\Delta X = X_+ - X_-$ is the sum of two optional processes, hence optional itself. Fix $\omega \in \Omega$. Consider the trajectory $f := X_\bullet(\omega)$. Due to statement (ii) of Theorem 2.6, there exists a finitely layered partition of $J(f) \subseteq L(f) = (0, \infty)$ which now is randomised, and it follows that we may write $J(f)$ as

$$J(f) = \dot{\bigcup_{m=1}^{\infty}} D_m(\omega),$$

where $\kappa_m(\omega) := \mathrm{card}(D_m(\omega)) < +\infty$ for all $m \in \mathbb{N}$. Let $\mathbb{M}(\omega) := \{m \in \mathbb{N} : D_m(\omega) \neq \emptyset\}$. Fix an arbitrary $m \in \mathbb{M}(\omega)$. Consider

$$0 < S_1^{(m)}(\omega) := \min(D_m(\omega))$$

and, if $\kappa_m(\omega) \ge 2$,

$$0 < S_{n+1}^{(m)}(\omega) := \min\big(D_m(\omega) \cap (S_n^{(m)}(\omega), \infty)\big),$$

where $n \in \{1, 2, \ldots, \kappa_m(\omega) - 1\}$. Since $\Delta X$ is optional, it follows that $\{\Delta X \in B\}$ is optional for all Borel sets $B \in \mathcal{B}(\mathbb{R})$. Moreover, since $\Delta f(0) = \Delta X_0(\omega) := 0$ (by assumption), it actually follows that $\{s \in \mathbb{R}_+ : (\omega, s) \in \{\Delta X \in C\}\} = \{s \in (0, \infty) : (\omega, s) \in \{\Delta X \in C\}\}$ for all Borel sets $C \in \mathcal{B}(\mathbb{R})$ which do not contain 0. Hence, as the construction of the sets $D_m(\omega)$ in the proof of Theorem 2.6 clearly shows, $S_1^{(m)}$ is the début of an optional set. Consequently, due to Theorem 1.2, it follows that $S_1^{(m)}$ is a stopping time. If $S_n^{(m)}$ is a stopping time, the stochastic interval $]\!]S_n^{(m)}, \infty[\![$ is optional too (cf. [8], Theorem 3.16). Thus, by construction, $S_{n+1}^{(m)}$ is the début of an optional set and hence a stopping time. Due to Lemma 4.2, we have

$$J(f) = \dot{\bigcup_{m \in \mathbb{M}(\omega)}} D_m(\omega) = \dot{\bigcup_{m \in \mathbb{M}(\omega)}} \dot{\bigcup_{n=1}^{\kappa_m(\omega)}} \{S_n^{(m)}(\omega)\}. \tag{4.2}$$

Hence, since for each trajectory of $X$ its set of jumps is not finite, the at most countable set $\mathbb{M}(\omega)$ is not finite, hence countable, and a simple relabelling of the stopping times $S_n^{(m)}$ finishes the proof. $\square$

We will see now how the choice of finitely layered jump partitions enables a natural approach for recovering the jump measure of a càdlàg and adapted stochastic process, and how we can transfer the structure of the jump measure to more general classes of optional processes which need not necessarily to be right-continuous.

Proposition 4.4 *Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space such that $\mathbf{F}$ satisfies the usual conditions. Let $X : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be an optional process such that all trajectories of $X$ are regulated and $\Delta X_0 := 0$. Consider*

$$M_X(\omega) := \Big\{(s, \Delta X_s(\omega)) : s \in J(X_\bullet(\omega))\Big\},$$

*where $\omega \in \Omega$. Then*

$$\mathrm{card}(M_X(\omega) \cap G) = \sum_{s \in J(X_\bullet(\omega))} \mathbb{1}_G\big(s, \Delta X_s(\omega)\big) = \sum_{s>0} \mathbb{1}_G\big(s, \Delta X_s(\omega)\big) \mathbb{1}_{\{\Delta X \neq 0\}}(\omega, s)$$

*for all $(\omega, G) \in \Omega \times \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$.*

*Proof.* For simplicity reasons, we may assume that the set of jumps of each trajectory of $X$ is not finite (due to Theorem 3.6 and representation (4.2)).

Fix $(\omega, G) \in \Omega \times \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$. Consider $D_X(\omega) := J(X_\bullet(\omega)) \subseteq L(X_\bullet(\omega)) = (0, \infty)$. Theorem 2.6 implies that the random set

$$M_X(\omega) := \Big\{(s, \Delta X_s(\omega)) : s \in D_X(\omega)\Big\} = \mathrm{Gr}\Big(\Delta X_\bullet(\omega)|_{D_X(\omega)}\Big)$$

is at most countable. Put $j_X(\omega, G) := \mathrm{card}(M_X(\omega) \cap G)$. Due to Theorem 4.3, it follows that there exists a sequence of stopping times $\{T_n : n \in \mathbb{N}\}$ such that

$$M_X(\omega) \cap G = \dot{\bigcup_{m=1}^{\infty}} \big\{\big(T_m(\omega), \Delta X_{T_m}(\omega)\big)\big\} \cap G,$$

where $\Delta X_{T_m}(\omega) := \Delta X_{T_m(\omega)}(\omega) = \Delta X(\omega, T_m(\omega))$. Since all unions are disjoint ones, the respective cardinals are additive. Consequently,

$$j_X(\omega, G) = \sum_{m=1}^{\infty} \mathrm{card}\big(\big\{\big(T_m(\omega), \Delta X_{T_m}(\omega)\big)\big\} \cap G\big) = \sum_{m=1}^{\infty} \mathbb{1}_G\big(T_m(\omega), \Delta X_{T_m}(\omega)\big),$$

and Theorem 3.6 together with Lemma 3.3 imply that

$$j_X(\omega, G) = \sum_{s \in D_X(\omega)} \mathbb{1}_G\big(s, \Delta X_s(\omega)\big) = \sum_{s>0} \mathbb{1}_G\big(s, \Delta X_s(\omega)\big) \mathbb{1}_{\{\Delta X \neq 0\}}(\omega, s).$$

□

THEOREM 4.5 *Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space such that $\mathbf{F}$ satisfies the usual conditions. Let $X : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be an optional process such that all trajectories of $X$ are regulated and $\Delta X_0 := 0$. Then the function*

$$\begin{array}{rcl} j_X : \Omega \times \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}) & \longrightarrow & \mathbb{Z}_+ \cup \{+\infty\} \\ (\omega, G) & \mapsto & \displaystyle\sum_{s>0} \mathbb{1}_G\big(s, \Delta X_s(\omega)\big) \mathbb{1}_{\{\Delta X \neq 0\}}(\omega, s) \end{array}$$

*is an integer-valued random measure.*

*Proof.* We only have to combine Theorem 4.3 and [8], Theorem 11.13. □

Put $G := B \times \Lambda$, where $B \in \mathcal{B}(\mathbb{R}_+)$ and $\Lambda \in \mathcal{B}(\mathbb{R})$. Since each trajectory of $X$ is regulated, $B \setminus \{0\} \subseteq (0, \infty) = L(X_\bullet(\omega))$. Hence, Lemma 3.3 directly leads to the following representation:

$$j_X(\omega, B \times \Lambda) = j_X(\omega, (B \setminus \{0\}) \times \Lambda) = \sum_{s \in B \setminus \{0\}} \mathbb{1}_\Lambda(\Delta X_s(\omega)) \mathbb{1}_{\{\Delta X \neq 0\}}(\omega, s)$$

for all $B \in \mathcal{B}(\mathbb{R}_+)$ and $\Lambda \in \mathcal{B}(\mathbb{R})$.

To sum up, $j_X(\omega, G)$ in general counts, $\omega$-by-$\omega$, the number of all $s > 0$ such that $\Delta X_s(\omega) \neq 0$ and $(s, \Delta X_s(\omega)) \in G$. In other words,

$$j_X(\omega, (\mathrm{d}t, \mathrm{d}x)) = \sum_{s>0} \mathbb{1}_{\{\Delta X \neq 0\}}(\omega, s) \cdot \delta_{\left(s, \Delta X_s(\omega)\right)}(\mathrm{d}t, \mathrm{d}x).$$

We finish this paper by considering right-continuous trajectories again and note the following

Corollary 4.6 *Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space such that $\mathbf{F}$ satisfies the usual conditions. If $X : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is an adapted and càdlàg process such that $\Delta X_0 := 0$, then $j_X$ is the jump measure of $X$.*

**Acknowledgements**: The author gratefully thanks Dave Applebaum and Finbarr Holland for highly fruitful discussions including Finbarr Holland's information about the very useful M. Sc. Thesis [13].

## References

[1] D. Applebaum. *Lévy Processes and Stochastic Calculus.* Cambridge Studies in Advanced Mathematics, Cambridge University Press (2004).

[2] R. F. Bass. *Probabilistic Techniques in Analysis.* Probability and its Applications, Springer, New York (1995).

[3] R. F. Bass. *General Theory of Processes.* http://www.math.uconn.edu/~bass/gtp.pdf.

[4] F. Burk. *Lebesgue Measure and Integration. An Introduction.* John Wiley & Sons, New York (1998).

[5] K. L. Chung and R. J. Williams. *Introduction to Stochastic Integration.* Second edition; Birkhäuser, Boston (1990).

[6] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel. Chapitres V á VIII. Théorie des martingales.* Revised edition; Hermann, Paris (1980).

[7] J. Dieudonné. *Foundations of Modern Analysis.* Pure and Applied Mathematics, Academic Press, New York and London (1960).

[8] S. He, J. Wang and J. Yan. *Semimartingales and Stochastic Calculus.* Science Press, Beijing and CRC Press Inc., Boca Raton (1992).

[9] J. Jacod. *Calcul stochastique et problèmes de martingales.* Lecture Notes in Mathematics 714, Springer, Berlin (1979).

[10] J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes.* Second edition; Grundlehren der Mathematischen Wissenschaften 288, Springer, Berlin (2003).

[11] F. C. Klebaner. *Introduction to Stochastic Calculus with Applications.* Imperial College Press, London (1998).

[12] M. Métevier. *Semimartingales. A Course on Stochastic Processes.* Walter de Gruyter, Berlin - New York (1982).

[13] J. O'Donovan. *Regulated Functions on Davison Spaces.* M. Sc. Thesis, National University of Ireland, Cork (2004).

[14] L. T. Nielsen, *Pricing and Hedging of Derivative Securities.* Oxford University Press, (1999).

[15] K. R. Parthasarathy *Probability Measures on Metric Spaces.* Probability and Mathematical Statistics, Academic Press, New York and London (1967).

[16] M. H. Protter and C. B. Morrey. *A First Course in Real Analysis.* Second edition; Springer, New York (1991).

[17] P. Protter. *Stochastic Integration and Differential Equations.* Second edition; Springer, Berlin (2004).